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EXTENSION-ORTHOGONAL COMPONENTS OF PREPROJECTIVE VARIETIES

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Abstract. Let Q be a Dynkin quiver, and let Λ be the corresponding preprojective algebra. Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a set of pairwise different indecomposable irreducible components of varieties of Λ-modules such that generically there are no extensions between C_i and C_j for all i, j. We show that the number of elements in $\mathcal{C}$ is at most the number of positive roots of Q. Furthermore, we give a module-theoretic interpretation of Leclerc's counterexample to a conjecture of Berenstein and Zelevinsky.

1. Introduction

Let k be an algebraically closed field. For a finitely generated k-algebra A let $\operatorname{mod}_A(\mathbf{d})$ be the affine variety of (left) A-modules with dimension vector $\mathbf{d}$. Throughout, we only consider finite-dimensional modules.

For irreducible components $C_1 \subseteq \operatorname{mod}_A(\mathbf{d}_1)$ and $C_2 \subseteq \operatorname{mod}_A(\mathbf{d}_2)$, define

$$\operatorname{ext}^1_A(C_1, C_2) = \min\{\dim \operatorname{Ext}^1_A(M_1, M_2) \mid (M_1, M_2) \in C_1 \times C_2\}.$$

An irreducible component $C \subseteq \operatorname{mod}_A(\mathbf{d})$ is *indecomposable* if it contains a dense subset of indecomposable A-modules. A general theory of irreducible components and their decomposition into indecomposable irreducible components was developed in [5]. Our aim is to apply this to the preprojective varieties.

If not mentioned otherwise, we always assume that Q is a Dynkin quiver of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. By R^+ we denote the set of positive roots of Q, and by Λ we denote the preprojective algebra associated to Q; see [20]. Let n be the number of vertices of Q, and let $\Lambda(\mathbf{d}) = \operatorname{mod}_\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$, be the variety of Λ-modules with dimension vector $\mathbf{d}$. The varieties $\Lambda(\mathbf{d})$ are called *preprojective varieties.* Since we consider only preprojective algebras of Dynkin type, the preprojective varieties coincide with the nilpotent varieties defined by Lusztig. We refer to [16, Section 12] for basic properties. Our main result is the following:

Theorem 1.1. *If $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a set of pairwise different indecomposable irreducible components such that $\operatorname{ext}^1_\Lambda(C_i, C_j) = 0$ for all $i, j \in I$, then $|I| \leq |R^+|$.*

The upper bound given in the theorem seems to be optimal. For example, if Q is of type $\mathbb{A}_n$, then it is easy to construct sets $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ of pairwise

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different indecomposable irreducible components such that $\mathrm{ext}^1_\Lambda(C_i, C_j) = 0$ for all $i, j \in I$ and $|I| = |R^+|$; see the end of Section 4.

As a consequence of the above theorem we get the following result on Λ-modules without self-extensions:

Corollary 1.2. *Let M be a Λ-module with $\mathrm{Ext}^1_\Lambda(M, M) = 0$. Then the number of pairwise non-isomorphic indecomposable direct summands of M is at most $|R^+|$.*

Let U_v^- be the negative part of the quantized enveloping algebra of the Lie algebra corresponding to Q. We regard U_v^- as a $\mathbb{Q}(v)$-algebra. Let $\mathcal{B}$ be the canonical basis and $\mathcal{B}^*$ the dual canonical basis of U_v^-; see [2], [14], [16] or [18] for definitions. By [12, Section 5] and [15] the elements of $\mathcal{B}$ (and thus of $\mathcal{B}^*$) correspond to the irreducible components of the preprojective varieties $\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$. Let $b^*(C)$ be the dual canonical basis vector corresponding to an irreducible component C. We denote the structure constants of U_v^- with respect to the basis $\mathcal{B}^*$ by $\lambda^E_{C,D}$, i.e.

$$b^*(C)b^*(D) = \sum_E \lambda^E_{C,D} b^*(E).$$

Following the terminology in [2], two dual canonical basis vectors $b^*(C)$ and $b^*(D)$ are called *multiplicative* if

$$b^*(C)b^*(D) = \lambda b^*(E)$$

for some irreducible component E and some $0 \neq \lambda \in \mathbb{Q}(v)$. One calls $b^*(C)$ and $b^*(D)$ *quasi-commutative* if

$$b^*(C)b^*(D) = \lambda b^*(D)b^*(C)$$

for some $0 \neq \lambda \in \mathbb{Q}(v)$. The following conjecture was stated in [2, Section 1]:

Conjecture 1.3 (Berenstein, Zelevinsky)**.** *Two dual canonical basis vectors are multiplicative if and only if they are quasi-commutative.*

One direction of this conjecture was proved by Reineke [18, Corollary 4.5]. The other direction turned out to be wrong. Namely, Leclerc [13] constructed examples of quasi-commutative elements in $\mathcal{B}^*$ which are not multiplicative. Using preprojective algebras, we give a module-theoretic interpretation of one of his examples.

Marsh and Reineke [17] conjectured that the multiplicative behaviour of dual canonical basis vectors should be related to sets of irreducible components with Ext^1 vanishing generically between them. This was the principal motivation for our work.

The paper is organized as follows: In Section 2 we review the main results from [5]. In Section 3 we recall known results for the case that Λ is an algebra of finite or tame representation type. The proof of Theorem 1.1 and its corollary can be found in Section 4. Finally, Section 5 is devoted to the interpretation of Leclerc's example.

2. Varieties of modules—Definitions and known results

In this section, we work with arbitrary finite quivers.

2.1. Let $Q = (Q_0, Q_1)$ be a finite quiver, where Q_0 denotes the set of vertices and Q_1 the set of arrows of Q. Assume that $|Q_0| = n$. For an arrow α let $s\alpha$ be its starting vertex and $e\alpha$ its end vertex. An element $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^n$ is called a *dimension vector* for Q. A *representation* of Q with dimension vector $\mathbf{d}$ is a matrix tuple $M = (M_\alpha)_{\alpha \in Q_1}$ with $M_\alpha \in \mathrm{M}_{d_{e\alpha} \times d_{s\alpha}}(k)$. A *path* of length $l \geq 1$ in Q is a sequence $p = \alpha_1 \cdots \alpha_l$ of arrows in Q_1 such that $s\alpha_i = e\alpha_{i+1}$ for $1 \leq i \leq l-1$. Define $sp = s\alpha_l$ and $ep = e\alpha_1$. For a representation M and a path $p = \alpha_1 \cdots \alpha_l$ define $M_p = M_{\alpha_1} \cdots M_{\alpha_l}$, which is a matrix in $\mathrm{M}_{d_{ep} \times d_{sp}}(k)$. A *relation* for Q is a k-linear combination $\sum_{i=1}^{t} \lambda_i p_i$ of paths p_i of length at least two such that $sp_i = sp_j$ and $ep_i = ep_j$ for all $1 \leq i, j \leq t$. A representation M satisfies such a relation if $\sum_{i=1}^{t} \lambda_i M_{p_i} = 0$. Given a set ρ of relations for Q, let $\mathrm{rep}_{(Q,\rho)}(\mathbf{d})$ be the affine variety of representations of Q with dimension vector $\mathbf{d}$ which satisfy all relations in ρ.

2.2. One can interpret this construction in a module-theoretic way. Namely, let kQ be the path algebra of Q, and let $A = kQ/(\rho)$, where (ρ) is the ideal generated by the elements in ρ. Then $\mathrm{mod}_A(\mathbf{d}) = \mathrm{rep}_{(Q,\rho)}(\mathbf{d})$ is the affine *variety of A-modules* with dimension vector $\mathbf{d}$. If $A = kQ/(\rho)$ is finite-dimensional, then A is a basic algebra. In this case, the vertices of Q correspond to the isomorphism classes of simple A-modules. The entry d_i, $i \in Q_0$, of $\mathbf{d}$ is the multiplicity of the simple module corresponding to i in a composition series of any $M \in \mathrm{mod}_A(\mathbf{d})$. The group $\mathrm{GL}(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(k)$ acts on $\mathrm{mod}_A(\mathbf{d})$ by conjugation, i.e.

$$g \cdot M = (g_{e\alpha} M_\alpha g_{s\alpha}^{-1})_{\alpha \in Q_1}.$$

The orbit of M under this action is denoted by $\mathcal{O}(M)$. There is a 1-1 correspondence between the set of orbits in $\mathrm{mod}_A(\mathbf{d})$ and the set of isomorphism classes of A-modules with dimension vector $\mathbf{d}$. For further details on varieties of modules, in particular on the close relation between representations of bounded quivers and modules over finite-dimensional algebras, we refer to [3].

2.3. Given irreducible components $C_i \subseteq \mathrm{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$, we consider all A-modules with dimension vector $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$, which are isomorphic to $M_1 \oplus \cdots \oplus M_t$, where $M_i \in C_i$ for all i. By

$$C_1 \oplus \cdots \oplus C_t$$

we denote the corresponding subset of $\mathrm{mod}_A(\mathbf{d})$. This is the image of the map

$$\mathrm{GL}(\mathbf{d}) \times C_1 \times \cdots \times C_t \longrightarrow \mathrm{mod}_A(\mathbf{d}),$$
$$(g, M_1, \cdots, M_t) \mapsto g \cdot \left(\bigoplus_{i=1}^{t} M_i \right).$$

We call $C_1 \oplus \cdots \oplus C_t$ the *direct sum* of the components C_i. It follows that the closure $\overline{C_1 \oplus \cdots \oplus C_t}$ is irreducible. For an irreducible component C, define $C^n = \bigoplus_{i=1}^{n} C$. We call C *indecomposable* if C contains a dense subset of indecomposable A-modules. The following result from [5] is an analogue of the Krull-Remak-Schmidt Theorem.

Theorem 2.1. *If $C \subseteq \mathrm{mod}_A(\mathbf{d})$ is an irreducible component, then*

$$C = \overline{C_1 \oplus \cdots \oplus C_t}$$

for some indecomposable irreducible components $C_i \subseteq \mathrm{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$. The components $C_1, \cdots, C_t$ are uniquely determined by this, up to reordering. The above direct sum is called the canonical decomposition of C.

However, the closure of a direct sum of irreducible components is not in general an irreducible component. The next result is also proved in [5].

Theorem 2.2. *If $C_i \subseteq \mathrm{mod}_A(\mathbf{d}_i)$, $1 \leq i \leq t$, are irreducible components, and $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of $\mathrm{mod}_A(\mathbf{d})$ if and only if $\mathrm{ext}^1_A(C_i, C_j) = 0$ for all $i \neq j$.*

Instead of taking direct sums of the modules in two irreducible components, one can take extensions. Let $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$ be dimension vectors, let $G = \mathrm{GL}(\mathbf{d}_1) \times \mathrm{GL}(\mathbf{d}_2)$, and let S be a G-stable subset of $\mathrm{mod}_A(\mathbf{d}_1) \times \mathrm{mod}_A(\mathbf{d}_2)$. We denote by $\mathcal{E}(S)$ the $\mathrm{GL}(\mathbf{d})$-stable subset of $\mathrm{mod}_A(\mathbf{d})$ corresponding to all modules M which belong to a short exact sequence

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

with $(M_1, M_2) \in S$; see [5] for more details.

For an irreducible component $C \subseteq \mathrm{mod}_A(\mathbf{d})$ let

$$\mu_g(C) = \dim C - \max\{\dim \mathcal{O}(M) \mid M \in C\}$$

be the *generic number of parameters of* C. Thus $\mu_g(C) = 0$ if and only if C contains a dense orbit $\mathcal{O}(M)$. For example, if P is a projective A-module, then $\mathrm{Ext}^1_A(P, P) = 0$. This implies that the closure of the orbit $\mathcal{O}(P)$ is an irreducible component, and we get $\mu_g(\overline{\mathcal{O}(P)}) = 0$. Also, if $C = \overline{C_1 \oplus \cdots \oplus C_t}$, then

$$\mu_g(C) = \sum_{i=1}^{t} \mu_g(C_i).$$

3. The finite and tame cases

Let A be a finite-dimensional k-algebra. Then A is called *representation-finite* if there are only finitely many isomorphism classes of indecomposable A-modules. The algebra A is *tame* if A is not representation-finite, and if for all dimension vectors $\mathbf{d}$ the indecomposable A-modules in $\mathrm{mod}_A(\mathbf{d})$ can be parametrized by a finite number of affine lines. Otherwise A is called *wild*. For precise definitions we refer to [4, Section 5.3]. The preprojective algebras of Dynkin type are selfinjective. We refer to [9] for the theory of representation-finite selfinjective algebras. The general theory of arbitrary representation-finite algebras is explained in [8]. Introductions to Auslander-Reiten theory and the representation theory of finite-dimensional algebras can be found in [1] and [19].

Proposition 3.1. *Let Q be a Dynkin quiver, and let Λ be the associated preprojective algebra. Then the following hold:*

- *Λ is representation-finite if and only if Q is of type $\mathbb{A}_i$, $i \leq 4$;*
- *Λ is tame if and only if Q is of type $\mathbb{A}_5$ or $\mathbb{D}_4$.*

The above proposition is well known to the experts. Let us sketch a proof: There always exists a simply connected Galois covering $F : \widetilde{\Lambda} \to \Lambda$ of Λ. In the cases $\mathbb{A}_i$, $i \leq 4$, one can construct all indecomposable $\widetilde{\Lambda}$-modules via the knitting procedure of preprojective components. One gets that $\widetilde{\Lambda}$ is locally representation-finite; thus Λ is representation-finite in these cases. For $\mathbb{A}_5$ and $\mathbb{D}_4$ the algebra $\widetilde{\Lambda}$ is the repetitive algebra of a tubular algebra. It follows that the push-down functor $\mathrm{mod}(\widetilde{\Lambda}) \to \mathrm{mod}(\Lambda)$ is dense, and that Λ is tame in these two cases. We refer to [6, Section 6] and [11] for more details. In all other cases, one can show that the algebra $\widetilde{\Lambda}$ contains wild full convex subalgebras; thus also Λ is wild. For the basics of covering theory we refer to [7] and [10].

If Λ is representation-finite, and if $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable irreducible components such that $\mathrm{ext}^1_\Lambda(C_i, C_j) = 0$ for all i, j, then $|I| = |R^+|$. This was proved by Marsh and Reineke, compare also [2].

For a tame algebra A one has $\mu_g(C) \leq 1$ for any indecomposable irreducible component $C \subseteq \mathrm{mod}_A(\mathbf{d})$. For Λ tame a complete classification of the indecomposable irreducible components, and a necessary and sufficient condition for $\mathrm{ext}^1_\Lambda(C, D) = 0$ for any two irreducible components C and D, were obtained in [11].

If Λ is of wild representation type, one should expect irreducible components C with $\mathrm{ext}^1_\Lambda(C, C) \neq 0$. Thus, maybe one should study sets $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ of irreducible components with the weaker condition $\mathrm{ext}^1_\Lambda(C_i, C_j) = 0$ for all $i \neq j$. However, we do not know how to generalize Theorem 1.1 to this case.

For the two tame cases, the following was proved in [11]:

Theorem 3.2. *Assume that Q is of type $\mathbb{A}_5$ or $\mathbb{D}_4$. Then the following hold:*

(1) *For any irreducible component $C \subseteq \Lambda(\mathbf{d})$ we have $\mathrm{ext}^1_\Lambda(C, C) = 0$.*
(2) *If $C \subseteq \Lambda(\mathbf{d})$ is an indecomposable irreducible component, then we have $\mu_g(C) = 0$ or $\mu_g(C) = 1$. For suitable $\mathbf{d}$ there exists an indecomposable irreducible component $C \subseteq \Lambda(\mathbf{d})$ with $\mu_g(C) = 1$.*
(3) *Let $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ be a maximal set of pairwise different indecomposable irreducible components such that $\mathrm{ext}^1_\Lambda(C_i, C_j) = 0$ for all i, j. Then there is at most one C_i with $\mu_g(C_i) = 1$. In this case, we have $|I| = |R^+| - 1$, and we get $|I| = |R^+|$, otherwise.*

This leads us to the following conjecture for arbitrary Dynkin quivers of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$:

Conjecture 3.3. *If $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable irreducible components such that $\mathrm{ext}^1_\Lambda(C_i, C_j) = 0$ for all i, j, then*

$$|I| = |R^+| - \sum_{i \in I} \mu_g(C_i).$$

4. Proof of Theorem 1.1

As before, let Q be a Dynkin quiver, and let

$$R^+ = \{\mathbf{a}_i \mid 1 \leq i \leq N\}$$

be the set of positive roots of Q. From now on $N = |R^+|$ will always be the number of positive roots of Q.

By Gabriel's Theorem there is a 1-1 correspondence between R^+ and the set of isomorphism classes of indecomposable kQ-modules. This correspondence associates to a root $\mathbf{a}_i$ the isomorphism class $[M(\mathbf{a}_i)]$ of an indecomposable kQ-module $M(\mathbf{a}_i)$ with dimension vector $\mathbf{a}_i$. By the Krull-Remak-Schmidt Theorem each kQ-module is isomorphic to a unique (up to reordering) direct sum of the indecomposable modules $M(\mathbf{a}_i)$. For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$ set

$$M_{\boldsymbol{\alpha}} = \bigoplus_{i=1}^{N} M(\mathbf{a}_i)^{\alpha_i} \text{ and } C_{\boldsymbol{\alpha}} = \overline{\pi_{\mathbf{d}}^{-1}(\mathcal{O}(M_{\boldsymbol{\alpha}}))},$$

where $\mathbf{d}$ is the dimension vector of $M_{\boldsymbol{\alpha}}$ and

$$\pi_{\mathbf{d}} : \Lambda(\mathbf{d}) \longrightarrow \mathrm{mod}_{kQ}(\mathbf{d})$$

is the canonical projection map. Let $\operatorname{Irr}(\Lambda)$ be the set of irreducible components of the varieties $\Lambda(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^n$. The three maps

$$\mathbb{N}^N \longrightarrow \{\mathcal{O}(M) \mid M \in \operatorname{mod}_{kQ}(\mathbf{d}), \mathbf{d} \in \mathbb{N}^n\} \longrightarrow \operatorname{Irr}(\Lambda) \longrightarrow \mathcal{B}^*,$$
$$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N) \mapsto \mathcal{O}(M_{\boldsymbol{\alpha}}) \mapsto C_{\boldsymbol{\alpha}} \mapsto b^*(C_{\boldsymbol{\alpha}}),$$

are all bijective.

Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^N$. By Theorem 2.2 the closure $\overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\beta}}}$ is an irreducible component if and only if $\operatorname{ext}^1_\Lambda(C_{\boldsymbol{\alpha}}, C_{\boldsymbol{\beta}}) = \operatorname{ext}^1_\Lambda(C_{\boldsymbol{\beta}}, C_{\boldsymbol{\alpha}}) = 0$. In this case, we have $\overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\beta}}} = C_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$.

Let $\boldsymbol{\delta}_j = (\delta_{1j}, \ldots, \delta_{Nj})$, $1 \leq j \leq N+1$, be non-zero pairwise different elements in $\mathbb{N}^N$ such that $C_{\boldsymbol{\delta}_j}$ is an indecomposable irreducible component for all j. To get a contradiction, we assume that $\operatorname{ext}^1_\Lambda(C_{\boldsymbol{\delta}_i}, C_{\boldsymbol{\delta}_j}) = 0$ for all $1 \leq i, j \leq N+1$. For $\mathbf{m} = (m_1, \cdots, m_{N+1}) \in \mathbb{N}^{N+1}$, define

$$C(\mathbf{m}) = C_{\Delta\mathbf{m}}, \text{ where } \Delta = (\delta_{ij}) \in \mathbb{N}^{N \times (N+1)}.$$

Thus $\boldsymbol{\delta}_j$ is the jth column of the matrix Δ. By our assumption we get

$$C(\mathbf{m}) = \overline{C_{\boldsymbol{\delta}_1}^{m_1} \oplus \cdots \oplus C_{\boldsymbol{\delta}_{N+1}}^{m_{N+1}}}.$$

Since the $C_{\boldsymbol{\delta}_j}$ are indecomposable, the above is the canonical decomposition of the irreducible component $C(\mathbf{m})$.

Now there exist some elements $\mathbf{m} = (m_1, \cdots, m_{N+1}) \neq \mathbf{l} = (l_1, \cdots, l_{N+1})$ in $\mathbb{N}^{N+1}$ such that

$$\Delta\mathbf{l} = \Delta\mathbf{m} \in \mathbb{N}^N.$$

This implies $C(\mathbf{m}) = C(\mathbf{l})$. Thus, we get a contradiction to the unicity of the canonical decomposition of irreducible components; see Theorem 2.1.

In fact, let $0 \neq \mathbf{z} \in \mathbb{Z}^{N+1}$ with $\Delta\mathbf{z} = 0$. Clearly, such a $\mathbf{z}$ always exists. Since all entries of Δ are non-negative, $\mathbf{z}$ has at least one negative entry. Let

$$l = -\min\{z_i \mid 1 \leq i \leq N+1\};$$

then trivially $\mathbf{l} = (l, l, \ldots, l)$ and $\mathbf{m} = \mathbf{l} + \mathbf{z}$ are as required. This finishes the proof of Theorem 1.1.

Corollary 1.2 follows immediately from the fact that an orbit $\mathcal{O}(N) \subseteq \operatorname{mod}_A(\mathbf{d})$ of an A-module N is open provided $\operatorname{Ext}^1_A(N, N) = 0$. Clearly, $\mathcal{O}(N)$ is open if and only if the closure $\overline{\mathcal{O}(N)}$ is an irreducible component. Then we use Theorems 1.1 and 2.2.

Finally, assume that Q is of type $\mathbb{A}_n$, and let $S_1, \ldots, S_n$ denote the (isomorphism classes of) simple Λ-modules. Set

$$H = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq n, 1 \leq j \leq n+1-i\},$$

and notice that $|H| = |R^+|$. For $(i,j) \in H$ there exists a unique (up to isomorphism) Λ-module $L_{(i,j)}$ with $\operatorname{top}(L_{(i,j)}) \cong S_i$ and $\operatorname{soc}(L_{(i,j)}) \cong S_j$ that admits S_1 as a composition factor. It is easy to check that $\operatorname{Ext}^1_\Lambda(L_{(i,j)}, L_{(p,q)}) = 0$ for all $(i,j), (p,q) \in H$. Thus

$$\mathcal{C} = \{\overline{\mathcal{O}(L_{(i,j)})} \mid (i,j) \in H\}$$

is a set of pairwise different indecomposable irreducible components with $\operatorname{ext}^1_\Lambda(C, D) = 0$ for all $C, D \in \mathcal{C}$, and we have $|\mathcal{C}| = |R^+|$.

5. Interpretation of Leclerc's example

In the following, we use the notation introduced at the beginning of Section 4. Reineke proved in [18, Lemma 4.6] that the multiplicativity of $b^*(C_{\boldsymbol{\alpha}})$ and $b^*(C_{\boldsymbol{\beta}})$ implies that

$$b^*(C_{\boldsymbol{\alpha}})b^*(C_{\boldsymbol{\beta}}) = v^m b^*(C_{\boldsymbol{\alpha}+\boldsymbol{\beta}})$$

for some $m \in \mathbb{Z}$. He also showed that $\lambda_{C,D}^E \neq 0$ if and only if $\lambda_{D,C}^E \neq 0$. This follows from [18, Proposition 4.4]. Thus one direction of Conjecture 1.3 holds. Namely, if two dual canonical basis vectors are multiplicative, then they are quasi-commutative. The following related problem should be of interest:

Problem 5.1. *Describe the elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^N$ such that*

$$C_{\boldsymbol{\alpha}+\boldsymbol{\beta}} = \overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\beta}}}.$$

As mentioned in the introduction, Leclerc recently constructed in [13] counterexamples for the other direction of the Berenstein-Zelevinsky Conjecture. We give a module-theoretic interpretation of one of his examples:

Let Q be the quiver of type $\mathbb{A}_5$ with arrows $a_i : i+1 \to i$, $1 \leq i \leq 4$. Thus Λ is given by the quiver

$$1 \underset{a_1}{\overset{\bar{a}_1}{\rightleftarrows}} 2 \underset{a_2}{\overset{\bar{a}_2}{\rightleftarrows}} 3 \underset{a_3}{\overset{\bar{a}_3}{\rightleftarrows}} 4 \underset{a_4}{\overset{\bar{a}_4}{\rightleftarrows}} 5$$

and the set of relations

$$\{a_1\bar{a}_1, \bar{a}_1a_1 - a_2\bar{a}_2, \bar{a}_2a_2 - a_3\bar{a}_3, \bar{a}_3a_3 - a_4\bar{a}_4, \bar{a}_4a_4\}.$$

Now R^+ contains exactly 15 elements, namely for each $1 \leq i \leq j \leq 5$ there is a positive root $[i,j] = (d_l)_{1 \leq l \leq 5}$ with $d_l = 1$ for $i \leq l \leq j$, and $d_l = 0$, otherwise. We identify $\mathbb{N}R^+$ with $\mathbb{N}^{15}$ by fixing a linear ordering on R^+, namely

$$\begin{aligned}&[1,1] < [1,2] < [1,3] < [1,4] < [1,5] < [2,2] < [2,3] < [2,4] < [2,5]\\ &< [3,3] < [3,4] < [3,5] < [4,4] < [4,5] < [5,5].\end{aligned}$$

Define

$$\begin{aligned}\boldsymbol{\alpha} &= [1,2] + [2,4] + [3,3] + [4,5],\\ \boldsymbol{\beta} &= [1,2] + [1,4] + [2,3] + [2,5] + [3,4] + [4,5].\end{aligned}$$

Thus, regarded as elements in $\mathbb{N}^{15}$,

$$\begin{aligned}\boldsymbol{\alpha} &= (0,1,0,0,0,0,0,1,0,1,0,0,0,1,0),\\ \boldsymbol{\beta} &= (0,1,0,1,0,0,1,0,1,0,1,0,0,1,0).\end{aligned}$$

In [13] Leclerc showed that

$$b^*(C_{\boldsymbol{\alpha}})^2 = v^{-2}\left(b^*(C_{\boldsymbol{\alpha}+\boldsymbol{\alpha}}) + b^*(C_{\boldsymbol{\beta}})\right).$$

This is obviously a counterexample to the Berenstein-Zelevinsky Conjecture. Now define

$$\begin{aligned}\boldsymbol{\beta}_1 &= [1,2] + [2,3] + [3,4] + [4,5],\\ \boldsymbol{\beta}_2 &= [1,4] + [2,5].\end{aligned}$$

Thus we have $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$.

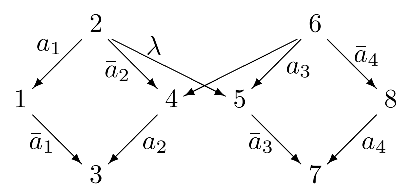
Proposition 5.2. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ be as above. Then the following hold:*

(1) *The irreducible components $C_{\boldsymbol{\alpha}}$, $C_{\boldsymbol{\beta}_1}$ and $C_{\boldsymbol{\beta}_2}$ are indecomposable with $\mu_g(C_{\boldsymbol{\alpha}}) = 1$ and $\mu_g(C_{\boldsymbol{\beta}_1}) = \mu_g(C_{\boldsymbol{\beta}_2}) = 0$.*

(2) *We have $C_{\boldsymbol{\alpha}+\boldsymbol{\alpha}} = \overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\alpha}}}$ and $C_{\boldsymbol{\beta}} = \overline{C_{\boldsymbol{\beta}_1} \oplus C_{\boldsymbol{\beta}_2}}$. Thus*

$$b^*(C_{\boldsymbol{\alpha}})^2 = v^{-2}\left(b^*(\overline{C_{\boldsymbol{\alpha}} \oplus C_{\boldsymbol{\alpha}}}) + b^*(\overline{C_{\boldsymbol{\beta}_1} \oplus C_{\boldsymbol{\beta}_2}})\right).$$

Proof. For $\lambda \in k \setminus \{0,1\}$ let M_λ be the 8-dimensional Λ-module where the arrows of Λ operate on a basis $\{1,\dots,8\}$ as in the following picture:



Thus, for example $a_1 \cdot 2 = 1$, $\bar{a}_2 \cdot 2 = 4 + \lambda 5$, $a_3 \cdot 6 = 4 + 5$, $\bar{a}_1 \cdot 1 = 3$, etc. Note that M_λ lies in $C_{\boldsymbol{\alpha}}$.

The modules M_λ are indecomposable and $\dim \operatorname{End}_\Lambda(M_\lambda) = 3$. It is known that the preprojective varieties $\Lambda(\mathbf{d})$ are equidimensional of dimension

$$\sum_{\alpha \in Q_1} d_{s\alpha} d_{e\alpha};$$

see for example [16, Section 12].

Thus each irreducible component of $\Lambda(1,2,2,2,1)$ has dimension $2+4+4+2 = 12$. The group $\mathrm{GL}(1,2,2,2,1)$ acts as described in Section 2 on $\Lambda(1,2,2,2,1)$ and has dimension 14. Thus we get $\dim \mathcal{O}(M_\lambda) = 14 - 3 = 11$. One checks easily that M_λ and M_μ are isomorphic if and only if $\lambda = \mu$. This implies

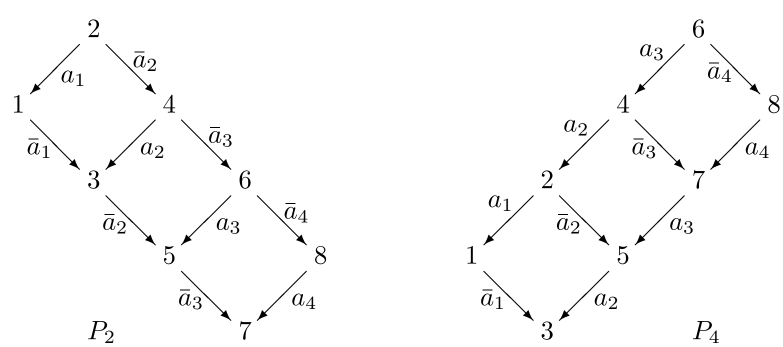
$$\dim \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0,1\}\}} = 11 + 1 = 12.$$

We get

$$C_{\boldsymbol{\alpha}} = \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0,1\}\}}.$$

Thus $C_{\boldsymbol{\alpha}}$ is an indecomposable irreducible component with $\mu_g(C_{\boldsymbol{\alpha}}) = 1$.

Next, let P_2 and P_4 be the indecomposable projective Λ-modules corresponding to the vertices 2 and 4, respectively. These modules are both 8-dimensional, and with the same convention as above we may describe them as follows:



We have $\mathrm{Ext}^1_\Lambda(P_i, P_j) = 0$ for all $i, j \in \{2, 4\}$. This follows directly from the projectivity of both modules. From this and the above pictures we get

$$\begin{aligned} C_{\boldsymbol{\beta}_1} &= \overline{\mathcal{O}(P_2)}, \\ C_{\boldsymbol{\beta}_2} &= \overline{\mathcal{O}(P_4)}, \\ C_{\boldsymbol{\beta}} &= \overline{C_{\boldsymbol{\beta}_1} \oplus C_{\boldsymbol{\beta}_2}}. \end{aligned}$$

In particular, $C_{\boldsymbol{\beta}_1}$ and $C_{\boldsymbol{\beta}_2}$ are indecomposable irreducible components with $\mu_g(C_{\boldsymbol{\beta}_1}) = \mu_g(C_{\boldsymbol{\beta}_2}) = 0$. This finishes the proof. □

For irreducible components $C \subseteq \Lambda(\mathbf{d})$ and $D \subseteq \Lambda(\mathbf{e})$, define

$$\mathcal{V}(C, D) = \bigcap_{U \subseteq C, V \subseteq D} \left\{ E \subseteq \Lambda(\mathbf{d} + \mathbf{e}) \text{ irred. comp.} \mid E \subseteq \overline{\mathcal{E}(U \times V)} \right\},$$

where U (resp. V) runs through all non-empty $\mathrm{GL}(\mathbf{d})$-stable (resp. $\mathrm{GL}(\mathbf{e})$-stable) open subsets of C (resp. D); see Section 2 for the definition of $\mathcal{E}(U \times V)$.

Using the previous proposition and some well-known results on the representation theory of the algebra Λ (see [11] and [19]), one can show that

$$\mathcal{V}(C_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = \{C_{\boldsymbol{\alpha}+\boldsymbol{\alpha}}, C_{\boldsymbol{\beta}}\}.$$

Note that $\mathrm{ext}^1_\Lambda(C_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = 0$, since $\mathrm{Ext}^1_\Lambda(M_\lambda, M_\mu) = 0$ for all $\lambda \neq \mu$. But one can show that $\dim \mathrm{Ext}^1_\Lambda(M_\lambda, M_\lambda) = 2$; see [11, Section 6]. For any M_λ there is a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow M_\lambda(2) \longrightarrow M_\lambda \longrightarrow 0,$$

where $M_\lambda(2)$ is the module of quasi-length two in the same Auslander-Reiten component as M_λ (it is known that M_λ lies in a homogeneous tube). In addition to this 'natural' self-extension, there exists a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow P_2 \oplus P_4 \longrightarrow M_\lambda \longrightarrow 0.$$

Motivated by our above analysis, one might conjecture the following:

Conjecture 5.3. (1) *If $E \in \mathcal{V}(C, D) \cup \mathcal{V}(D, C)$, then $\lambda^E_{C,D} \neq 0$.*

(2) *If irreducible components C and D contain non-empty stable open subsets $U \subseteq C$ and $V \subseteq D$ such that $\mathrm{Ext}^1_\Lambda(M, N) = 0$ for all $M \in U, N \in V$, then $b^*(C)$ and $b^*(D)$ are multiplicative.*

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